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The Characterization of the Derivatives for Linear Combinations of Post–Widder Operators in L_p

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The aim of the paper is to characterize the global rate of approximation of derivatives $f^{(l)}$ through corresponding derivatives of linear combinations of Post–Widder operators in an appropriate weighted L_p-metric using a weighted Ditzian and Totik modulus of smoothness, and also to characterize derivatives of these operators in Besov spaces of Ditzian–Totik type. © 1999 Academic Press

Key Words: Post–Widder operators; weighted simultaneous approximation; linear combination; Besov spaces of Ditzian–Totik type.

1. INTRODUCTION AND MAIN RESULTS

It is well known that Post–Widder operators constitute the real inversion formula for the Laplace transform. Post–Widder operators are given by

$$P_n(f, x) = \frac{(n/x)^n}{(n-1)!} \int_0^\infty e^{-nu/x} u^{n-1} f(u)\, du, \qquad x \in (0, \infty), \tag{1.1}$$

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where $f \in L_p[0, \infty)$ $(1 \leqslant p < \infty)$ or $f \in C[0, \infty)$. We will use for $P_n(f, x)$ the combination $P_{n,r}(f, x)$ given by [1, Chapter 9]

$$P_{n,r}(f, x) = \sum_{i=0}^{r-1} C_i(n)\, P_{n_i}(f, x), \qquad x \in (0, \infty), \quad r \in N, \tag{1.2}$$

where n_i and $C_i(n)$ satisfy

(a) $n = n_0 < \cdots < n_{r-1} \leqslant An$;
(b) $\sum_{i=0}^{r-1} |C_i(n)| \leqslant C$;
(c) $\sum_{i=0}^{r-1} C_i(n) = 1$;
(d) $\sum_{i=0}^{r-1} C_i(n)\, P_{n_i}((\cdot - x)^k, x) = 0, \quad 1 \leqslant k \leqslant r-1.$

Concerning the approximation by linear combinations of Post–Widder operators, Ditzian and Totik [1] proved direct and converse results for these operators in L_p. Their main theorems show, for $f \in L_p[0, \infty)$, $1 \leqslant p \leqslant \infty$ (with $C[0, \infty)$, for $p = \infty$) and $\varphi(x) = x$, that

$$\begin{aligned} \|P_{n,r}(f, x) - f(x)\|_p = \mathcal{O}(n^{-\alpha}) \Leftrightarrow \omega_\varphi^{2r}(f, t)_p \\ = \mathcal{O}(t^{2\alpha}) \qquad (0 < \alpha < r), \end{aligned} \tag{1.3}$$

where

$$\omega_\varphi^{2r}(f, t)_p = \sup_{0 < h \leqslant t} \|\Delta_{h\varphi}^{2r} f\|_{L_p[0, \infty)}, \qquad \varphi(x) = x, \quad f \in L_p[0, \infty), \tag{1.4}$$

and

$$\Delta_h^{2r} f(x) = \begin{cases} \sum_{j=0}^{2r} (-1)^j \binom{2r}{j} f(x + (r-j)\, h), & \text{for} \quad x \geqslant rh \\ 0, & \text{otherwise.} \end{cases}$$

The first aim of this paper is to prove new direct and converse results on weighted simultaneous approximation by the method of linear combinations of Post–Widder operators in L_p, $1 \leqslant p \leqslant \infty$. Our results are stated as follows.

THEOREM 1.1. *Let $f, f^{(l)}, \varphi^l f^{(l)} \in L_p[0, \infty)$, $1 \leqslant p \leqslant \infty$ (with $C[0, \infty)$, for $p = \infty$), $l \in N_0$, $r \in N$. Then*

$$\|\varphi^l (P_{n,r}(f, x) - f(x))^{(l)}\|_p \leqslant C\{\omega_\varphi^{2r}(f^{(l)}, n^{-1/2})_{\varphi^l, p} + n^{-r} \|\varphi^l f^{(l)}\|_p\}.$$

Here

$$\omega_\varphi^{2r}(f^{(l)}, t)_{\varphi^l, p} = \sup_{0 < h \leqslant t} \|\varphi^l \Delta_{h\varphi}^{2r} f^{(l)}\|_{L_p[0, \infty)}, \qquad f^{(l)}, \varphi^l f^{(l)} \in L_p[0, \infty)$$

is the weighted Ditzian–Totik modulus of smoothness which was shown in [1, Chap. 6] to be equivalent to the weighted K-functional defined by

$$K_\varphi^{2r}(f^{(l)}, t^{2r})_{\varphi^l, p} = \inf \{ \|\varphi^l (f^{(l)} - g)\|_p + t^{2r} \|\varphi^{l+2r} g^{(2r)}\|_p, \\ \varphi^l g, \varphi^{l+2r} g^{(2r)} \in L_p[0, \infty), 1 \leqslant p \leqslant \infty \}. \tag{1.5}$$

THEOREM 1.2. *Let $f, f^{(l)}, \varphi^l f^{(l)} \in L_p[0, \infty)$, $1 \leqslant p \leqslant \infty$ (with $C[0, \infty)$, for $p = \infty$), $l \in N_0$, $r \in N$, $t > 0$. Then*

$$K_\varphi^{2r}(f^{(l)}, t^{2r})_{\varphi^l, p} \leqslant \|\varphi^l (P_{n,r}(f, x) - f(x))^{(l)}\|_p + M(nt)^r K_\varphi^{2r}(f^{(l)}, n^{-2r})_{\varphi^l, p}. \tag{1.6}$$

From Theorems 1.1 and 1.2 and Corollary 6.3.2 in [1], and the Berens–Lorentz Lemma [1, Chap. 9], we obtain

THEOREM 1.3. *Let $f, f^{(l)}, \varphi^l f^{(l)} \in L_p[0, \infty)$, $1 \leqslant p \leqslant \infty$ (with $C[0, \infty)$, for $p = \infty$), $l \in N_0, r \in N, t > 0$, and $l/2 < \alpha < l/2 + r$. Then the following statements are equivalent.*

$$\|\varphi^l (P_n(f, r-1, x) - f(x))^{(l)}\|_p = \mathcal{O}(n^{l/2 - \alpha}); \tag{1.7}$$

$$\omega_\varphi^{2r}(f^{(l)}, t)_{\varphi^l, p} = \mathcal{O}(t^{2l - \alpha}); \tag{1.8}$$

$$\omega_\varphi^{2r+l}(f, t)_p = \mathcal{O}(t^{2\alpha}). \tag{1.9}$$

Remark 1.4. For $l = 0$, we obtain (1.3) mentioned above. Some ideas of our proof of Theorem 1.3 are from [5].

With the definition of (1.4), the Besov space of Ditzian–Totik type $B_q^\alpha(L_p[0, \infty))$ are defined for $0 < \alpha < m$, $1 \leqslant p \leqslant \infty$, and $0 < q \leqslant \infty$ as the set of all functions $f \in L_p[0, \infty)$ for which

$$|f|_{B_q^\alpha(L_p[0, \infty))} = \left(\int_0^\infty (t^{-\alpha} \omega_\varphi^m(f, t)_p)^q \frac{1}{t} \, dt \right)^{1/q} \tag{1.10}$$

is finite. Here, m is any integer larger than α. When $q = \infty$, the usual change from integral to sup is made in (1.10). We define the following norms or quasi-norms for $B_q^\alpha(L_p[0, \infty))$:

$$\|f\|_{B_q^\alpha(L_p[0, \infty))} = \|f\|_{L_p[0, \infty)} + |f|_{B_q^\alpha(L_p[0, \infty))}. \tag{1.11}$$

These Besov spaces were defined for $1 \leqslant q \leqslant \infty$ and studied by Zhou [8] and also studied by several other authors [3, 4].

We note that when $q < 1$, (1.11) is not really a norm, it is only a quasi-norm, and that different values of $m > \alpha$ result in norm or quasi-norm

(1.11) which are equivalent. This is proved by establishing inequalities between the modulus of smoothness $\omega_\varphi^m(f, t)_p$ and $\omega_\varphi^{m+1}(f, t)_p$. A simple inequality is $\omega_\varphi^{m+1}(f, t)_p \leqslant C\omega_\varphi^m(f, t)_p$, which follows from [1, Chap. 4]. In the other direction, we have the Marchaud type inequality

$$\omega_\varphi^m(f, t)_p \leqslant Ct^m \left\{ \int_t^c \frac{\omega_\varphi^{m+1}(f, u)_p}{u^{m+1}}\, du + \|f\|_p \right\},$$

which was also proved in [1, Chap. 4]. Using these two inequalities for modulus $\omega_\varphi^m(f, t)_p$ together with Hard inequality [1, Chap. 9], one shows that any two norms or quasi-norms given by (1.11) are equivalent provided that both m satisfy $m > \alpha$.

Some papers [2, 6, 7] have characterized smoothness of the functions in $C[0, 1]$ by derivatives of Bernstein-type integral operators and also in L_p, $1 \leqslant p \leqslant \infty$ by derivatives of Bernstein–Durrmeyer operators. In the second part of this paper, by using of the commutative property of these operators, we will show that the derivatives of $P_{n,r}(f, x)$ can also be characterized in the Besov spaces defined as in (1.11).

Theorem 1.5. *For* $r \in N$, $0 < \alpha < r$, $f \in L_p[0, \infty)$, $1 \leqslant p \leqslant \infty$ (*with* $C[0, \infty)$, *for* $p = \infty$), *and* $0 < q \leqslant \infty$, *the norms or quasi-norms*

$$\|f\|_{B_q^\alpha(L_p[0, \infty))}, \tag{1.12}$$

$$\|\{(n+1)^\alpha \|P_{n,r}(f, x) - f\|_p\}\|_{l_q^*} + \|f\|_p, \tag{1.13}$$

and

$$\|\{n^{\alpha - r} \|\varphi^{2r} P_{n,r}^{(2r)}(f, x)\|_p\}\|_{l_q^*} + \|f\|_p \tag{1.14}$$

are equivalent, where for a sequence $\{a_n\}_{n=1}^\infty$

$$\|\{a_n\}\|_{l_q^*} = \begin{cases} \left(\sum_{n=1}^\infty |a_n|^q\, n^{-1} \right)^{1/q}, & \text{for } 0 < q < \infty \\ \sup_n |a_n|, & q = \infty. \end{cases}$$

Remark 1.6. For $q = \infty$ and $0 < \alpha < r$, we obtain the equivalent relations

$$\begin{aligned} \|P_{n,r}(f, x) - f(x)\|_p = \mathcal{O}(n^{-\alpha}) &\Leftrightarrow \omega_\varphi^{2r}(f, t)_p \\ &= \mathcal{O}(t^{2\alpha}) \Leftrightarrow \|\varphi^{2r} P_n^{(2r)} f\|_p = \mathcal{O}(n^{r-\alpha}), \end{aligned}$$

which is also similar to results given in [2] for Bernstein–Durrmeyer operators and linear combinations of Bernstein–Durrmeyer operators.

Throughout this paper, M and C will always stand for positive constants which are dependent only on p, q, r, l, and α; their values may be different at different occurrences and $\varphi(x)=x$.

2. DEFINITIONS AND AUXILIARY RESULTS

For convenience we introduce the auxiliary operators given for $n \geqslant l+1$, $l \in N_0$, $f \in L_p[0, \infty)$ $(1 \leqslant p < \infty)$, or $f \in C[0, \infty)$ by

$$P_{n, r, l}(f, x) = \sum_{i=0}^{r-1} C_i(n)\, \bar{P}_{n_i, l}(f, x), \qquad x \in (0, \infty),$$

where

$$\begin{aligned} \bar{P}_{n_i, l}(f, x) &= \frac{n_i^{n_i}}{(n_i-1)!} \int_0^\infty e^{-n_i t} t^{n_i+l-1} f(tx)\, dt \\ &= \frac{n_i^{n_i}}{(n_i-1)!} \int_0^\infty e^{-n_i u/x} \frac{u^{n_i+l-1}}{x^{n_i+l}} f(u)\, du, \qquad x \in (0, \infty). \end{aligned} \tag{2.1}$$

It is easy to see that these operators too are bounded on the spaces $L_p[0, \infty)$, $1 \leqslant p \leqslant \infty$ (with $C[0, \infty)$, for $p=\infty$) and that the following four representations are also valid.

If f, $\varphi^l f \in L_p[0, \infty)$, $1 \leqslant p \leqslant \infty$ (with $C[0, \infty)$, for $p=\infty$), $l \in N_0$, then

$$\varphi^l P_{n, r, l}(f, x) = P_{n, r}(\varphi^l f, x). \tag{2.2}$$

If f, $f^{(l)} \in L_p[0, \infty)$, $1 \leqslant p \leqslant \infty$ (with $C[0, \infty)$, for $p=\infty$), $l \in N_0$, then

$$P_{n, r}^{(l)}(f, x) = P_{n, r, l}(f^{(l)}, x). \tag{2.3}$$

If f, $\varphi^l f \in L_p[0, \infty)$, $1 \leqslant p \leqslant \infty$ (with $C[0, \infty)$, for $p=\infty$), $l \in N_0$, then

$$\varphi^l P_{n, r}^{(l)}(f, x) = P_{n, r}(\varphi^l f^{(l)}, x). \tag{2.4}$$

If f, $\varphi^{2r} f^{(2r)} \in L_p[0, \infty)$, $1 \leqslant p \leqslant \infty$ (with $C[0, \infty)$, for $p=\infty$), $l \in N_0$, then

$$\varphi^{2r} P_{n, r, l}^{(2r)}(f, x) = P_{n, r}(\varphi^{2r} f^{(2r)}, x). \tag{2.5}$$

For the proofs of our main theorems, we will need the following lemmas, which are of importance by themselves.

LEMMA 2.1. *If* $f, \varphi^l f \in L_p[0, \infty)$, $1 \leqslant p \leqslant \infty$ (*with* $C[0, \infty)$, *for* $p = \infty$), $l \in N_0$, *then*

$$\|\varphi^l P_{n, rl}(f, x)\|_p \leqslant C \|\varphi^l f\|_p. \tag{2.6}$$

Proof. With (2.2), we have

$$\|\varphi^l P_{n, r, l}(f, x)\|_p = \|P_{n, r}(\varphi^l f, x)\|_p \leqslant C \|\varphi^l f\|_p.$$

LEMMA 2.2. *If* $f, \varphi^{2r} f^{(2r)}, \varphi^{2r+l} f^{(2r)} \in L_p[0, \infty)$, $1 \leqslant p \leqslant \infty$ (*with* $C[0, \infty)$, *for* $p = \infty$), $l \in N_0$, *then*

$$\|\varphi^{2r+l} P^{(2r)}_{n, r, l}(f, x)\|_p \leqslant C \|\varphi^{2r+l} f^{(2r)}\|_p. \tag{2.7}$$

Proof. Multiplying both sides in relation (2.5) by φ^l, we obtain

$$\varphi^{2r+l} P^{(2r)}_{n, r, l}(f, x) = \varphi^l P_{n, r, l}(\varphi^{2r} f^{(2r)}, x).$$

Using (2.2), we have

$$\varphi^{2r+l} P^{(2r)}_{n, r, l}(f, x) = P_{n, r}(\varphi^{2r+l} f^{(2r)}, x),$$

which implies

$$\|\varphi^{2r+l} P^{(2r)}_{n, r, l}(f, x)\|_p \leqslant C \|\varphi^{2r+l} f^{(2r)}\|_p.$$

LEMMA 2.3. *If* $f, \varphi^l f \in L_p[0, \infty)$, $1 \leqslant p \leqslant \infty$ (*with* $C[0, \infty)$, *for* $p = \infty$), $l \in N_0$, *then*

$$\|\varphi^{2r+l} P^{(2r)}_{n, r, l}(f, x)\|_p \leqslant C n^r \|\varphi^l f\|_p. \tag{2.8}$$

Proof. By simple computation, we have

$$\bar{P}_{n_i, l}(f, x) = \frac{1}{x^l} P_{n_i}(\varphi^l f, x), \tag{2.9}$$

then

$$\varphi^{2r+l} \bar{P}^{(2r)}_{n_i, l}(f, x) = \sum_{j=0}^{2r} C_{r, j, l} \varphi(x)^{2r-j} P^{(2r-j)}_{n_i}(\varphi^l f, x).$$

From [1, Chap. 9], it is easy to obtain

$$P^{(2r-j)}_{n_i}(\varphi^l f, x) = \sum_{v=0}^{2r-j} Q_v(n_i, x) P_{n_i}((\cdot - x)^v \varphi^l f, x),$$

where $Q_v(n_i, x) = \sum_{2s+r-v=2r-j} C(v, \tau)\, n_i^s / x^{2s+\tau}$. Therefore

$$\varphi(x)^{2r-j}\, |Q_v(n_i, x)| \leqslant C \frac{n^{r+v/2}}{x^v}, \qquad x > 0, \quad v = 0, 1, 2, 3, ...0, 2r.$$

Thus, following the proof of Lemma 9.4.1 in [1], it is easy to complete the proof of Lemma 2.3.

LEMMA 2.4. *If* $f, \varphi^l f, \varphi^{2r+l} f^{(2r)} \in L_p[0, \infty)$, $1 \leqslant p \leqslant \infty$ *(with* $C[0, \infty)$, *for* $p = \infty$), $l \in N_0$, *then*

$$\|\varphi^l (P_{n,r,l}(f, x) - f(x))\|_p \leqslant C n^{-r} \{ \|\varphi^l f\|_p + \|\varphi^{2r+l} f^{(2r)}\|_p \}. \tag{2.10}$$

Proof. With (2.2), we have

$$\varphi^l (P_{n,r,l}(f, x) - f(x)) = P_{n,r}(\varphi^l f, x) - (\varphi^l f)(x).$$

We expand $\varphi^l f$ by the Taylor formula

$$(\varphi^l f)(t) = \sum_{j=0}^{2r-1} \frac{(t-x)^j}{j!} (\varphi^l f)^{(j)}(x) + R_{2r}(\varphi^l f, t, x),$$

with the integral remainder

$$R_{2r}(\varphi^l f, t, x) = \frac{1}{(2r-1)!} \int_x^t (t-v)^{2r-1} (\varphi^l f)^{(2r)}(v)\, dv.$$

We write

$$\begin{aligned} P_{n,r}(\varphi^l f, x) &= \sum_{j=0}^{2r-1} \frac{1}{j!} (\varphi^l f)^{(j)}(x)\, P_{n,r}((\cdot - x)^j, x) \\ &\quad + P_{n,r}\left(\frac{1}{(2r-1)!} \int_x^t (t-v)^{2r-1} (\varphi^l f)^{(2r)}(v)\, dv, x \right). \end{aligned}$$

It follows from the definition of $P_{n,r}(f, x)$ that

$$\begin{aligned} P_{n,r}(\varphi^l f, x) - (\varphi^l f)(x) &= \sum_{j=r}^{2r-1} \frac{1}{j!} (\varphi^l f)^{(j)}(x)\, P_{n,r}((\cdot - x)^j, x) \\ &\quad + P_{n,r}\left(\frac{1}{(2r-1)!} \int_x^t (t-v)^{2r-1} (\varphi^l f)^{(2r)}(v)\, dv, x \right) \\ &= I_1 + I_2. \end{aligned}$$

Using Lemma 9.5.1 of [1], we have

$$|I_1| \leqslant Mn^{-r} \sum_{j=r}^{2r-1} \frac{1}{j!} |(\varphi^l f)^{(j)}(x)\, \varphi(x)^j|$$
$$\leqslant Mn^{-r} \sum_{j=r}^{2r-1} \sum_{s=0}^{j} |\varphi^{l-s+j}(x)\, f^{(j-s)}(x)|.$$

Then

$$\|I_1\|_p \leqslant Mn^{-r} \left(\sum_{j=r}^{2r-1} \sum_{s=0}^{j-1} \|^{l-s+j} f^{(j-s)}\|_p + \|\varphi^l f\|_p \right).$$

Using that $\|\varphi^{l+j-s} f^{(j-s)}\|_p \leqslant C \|\varphi^{l+j-s+1} f^{(j-s+1)}\|_p$, for $j-s=1, 2,3,...$ $2r-1$, which follows easily from the Hard inequality for $1 \leqslant p < \infty$, [1, Chap. 9] and for $p=\infty$, $\|\varphi^{l+j-s} f^{(j-s)}\|_\infty = \|\varphi^{l+j-s} \int_x^\infty f^{(j-s+1)}(t)\, dt\|_\infty$ $\leqslant C \|\varphi^{l+j-s+1} f^{(j-s+1)}\|_\infty$, we see that

$$\|I_1\|_p \leqslant Mn^{-r} (\|\varphi^{2r+l} f^{(2r)}\|_p + \|\varphi^l f\|_p).$$

To prove (2.10), it remains to show that for $1 \leqslant p \leqslant \infty$

$$\|I_2\|_p \leqslant Mn^{-r} (\|\varphi^{2r+l} f^{2r)}\|_p + \|\varphi^l f\|_p).$$

From [1, Lemma 9.5.2], we have

$$\|I_2\|_p \leqslant Mn^{-r} (\|\varphi^{2r} (\varphi^l f)^{(2r)}\|_p + \|\varphi^l f\|_p)$$
$$\leqslant Mn^{-r} \left(\sum_{j=0}^{2r-1} C_{j,l,r} \|\varphi^{2r+l-j} f^{(2r-j)}\|_p + \|\varphi^l f\|_p \right),$$

thus we obtain in a way similar to before the estimate

$$\|I_2\|_p \leqslant Mn^{-r} (\|\varphi^{2r+l} f^{(2r)}\|_p + \|\varphi^l f\|_p).$$

The proof of Lemma 2.4 is complete.

Next we will prove the commutative property of these operators, which is important for our purpose.

LEMMA 2.5. *For $f(x) \in L_p[0, \infty)$, $1 \leqslant p \leqslant \infty$ (with $C[0, \infty)$, for $p=\infty$), $m, n = 1, 2, 3, ...,$ then*

$$P_{n,r}(P_{m,r}(f, \cdot), x) = P_{m,r}(P_{n,r}(f, \cdot), x), \qquad x > 0. \tag{2.11}$$

Proof. From the definition of linear combination for the Post–Widder operator $P_{n,r}(f, x)$, we need only to show that

$$P_n(P_m(f, \cdot), x) = P_m(P_n(f, \cdot), x), \qquad m, n = 1, 2, 3, ..., \quad x > 0.$$

For $p = \infty$,

$$P_n(P_m(f, \cdot), x) = \int_0^\infty \frac{(n/x)^n}{(n-1)!} e^{-nu/x} u^{n-1} P_m(f, u)\, du$$

$$= \int_0^\infty \frac{(n/x)^n}{(n-1)!} e^{-nu/x} u^{n-1} \int_0^\infty \frac{(m/u)^m}{(m-1)!} e^{-mv/u} v^{m-1} f(v)\, dv\, du.$$

Since

$$\int_0^\infty \frac{(n/x)^n}{(n-1)!} e^{-nu/x} u^{n-1} \int_0^\infty \frac{(m/u)^m}{(m-1)!} e^{-mv/u} v^{m-1} |f(v)|\, dv\, du \leqslant \|f\|_\infty,$$

thus, by using of Fubini's theorem, we have

$$P_n(P_m(f, \cdot), x) = \int_0^\infty f(v)\, v^{m-1}\, dv \int_0^\infty \frac{(n/x)^n}{(n-1)!} \times e^{-nu/x} u^{n-1} \frac{(m/u)^m}{(m-1)!} e^{-mv/u}\, du.$$

Let $v/u = w/x$, then

$$\int_0^\infty f(v)\, v^{m-1}\, dv \int_0^\infty \frac{(n/x)^n}{(n-1)!} e^{-nu/x} u^{n-1} \frac{(m/u)^m}{(m-1)!} e^{-mv/u}\, du$$

$$= \int_0^\infty f(v)\, v^{m-1}\, dv \int_0^\infty \frac{(n/x)^n}{(n-1)!} e^{-nv/w} \left(\frac{xv}{w}\right)^{n-1} \times \frac{1}{(m-1)!} \left(\frac{mw}{xv}\right)^m e^{-mw/x} \frac{xv}{w^2}\, dw$$

$$= \int_0^\infty f(v)\, v^{m-1}\, dv \int_0^\infty \frac{(m/x)^m}{(m-1)!} e^{-mw/x} w^{m-1} \frac{(n/w)^n}{(n-1)!} e^{-nv/w}\, dw$$

$$= \int_0^\infty \frac{(m/x)^m}{(m-1)!} e^{-mw/x} w^{m-1} P_n(f, w)\, dw$$

$$= P_m(P_n(f, \cdot), x).$$

Thus we prove Lemma 2.5 for $p=\infty$. For $1\leqslant p<\infty$, we define $C_0=\{f\in C[0,\infty),\ \operatorname{supp} f\subset[0,M] \text{ for some } M>0\}$. It is obvious that C_0 is dense in $L_p[0,\infty)$ for $1\leqslant p<\infty$; therefore we need only to prove the result for $f\in C_0$, which is similar to the case of $p=\infty$. Then the proof of Lemma 2.5 is complete.

3. PROOFS OF THE MAIN RESULTS

Proof of Theorem 1.1. The essential tool in this proof is the equivalence $\omega_\varphi^{2r}(f^{(l)},t)_{\varphi^l,p}\sim K_\varphi^{2r}(f^{(l)},t^{2r})_{\varphi^l,p}$ and it has to be shown that

$$\|\varphi^l(P_{n,r}(f,x)-f(x))^{(l)}\|_p\leqslant C\{K_\varphi^{2r}(f^{(l)},n^{-r})_{\varphi^l,p}+n^{-r}\|\varphi^l f^{(l)}\|_p\}. \qquad (3.1)$$

Now for every $g\in L_p[0,\infty)$ with $\varphi^l g,\ \varphi^{2r+l}g^{(2r)}\in L_p[0,\infty)$, Lemma 2.1, Lemma 2.4 and (2.3) imply

$$\begin{aligned}\|\varphi^l(P_{n,r}(f,x)-f(x))^{(l)}\|_p&\leqslant\|\varphi^l P_{n,r,l}(f^{(l)}-g,x)\|_p+\|\varphi^l(f^{(l)}(x)-g(x))\|_p\\&\quad+\|\varphi^l(P_{n,r,l}(g,x)-g(x))\|_p\\&\leqslant C\{\|\varphi^l(f^{(l)}(x)-g(x))\|_p\\&\quad+\|\varphi^l(P_{n,r,l}(g,x)-g(x))\|_p\}\\&\leqslant C\{\|\varphi^l(f^{(l)}(x)-g(x))\|_p\\&\quad+n^{-r}\|\varphi^{2r+l}g^{(2r)}\|_p+n^{-r}\|\varphi^l f\|_p\}.\end{aligned}$$

Taking here the infimum over all g subject to the definition of the weighted K-functional gives the desired inequality (3.1). Then we complete the proof of Theorem 1.1.

Proof of Theorem 1.2. From the definition of the weighted K-functional in (1.5), Lemma 2.2 and Lemma 2.3, it is easy to prove theorem 1.2.

Proof of Theorem 1.5. We only prove the equivalence between (1.12) and (1.14) for $0<q<\infty$, since the proof for the case $q=\infty$ is simple and the equivalence between (1.12) and (1.13) can be proved by using the same method as the proof of Theorem 4.1 of [3].

From [1, Chap. 9], it is easy to obtain

$$\|\varphi^{2r}P_{n,r}^{(2r)}(f,x)\|_p\leqslant Mn^r\omega_\varphi^{2r}(f,n^{-1/2})_p, \qquad 1\leqslant p\leqslant\infty.$$

Hence, for $0<q<\infty$

$$\begin{aligned}
&\sum_{n=1}^{\infty} (n^{\alpha-r}\,\|\varphi^{2r}P_{n,r}^{(2r)}(f,x)\|_p)^q\,\frac{1}{n}\\
&\quad\leqslant M\left(\sum_{n=2}^{\infty} n^{\alpha q-1}\omega_\varphi^{2r}(f,n^{-1/2})_p^q+\|f\|_p^q\right)\\
&\quad\leqslant M\left(\sum_{n=2}^{\infty}\int_{n^{-1/2}}^{(n-1)^{-1/2}}(t^{-2\alpha}\omega_\varphi^{2r}(f,t)_p)^q\,\frac{1}{t}\,dt+\|f\|_p^q\right)\\
&\quad\leqslant M\left(\int_0^\infty (t^{-2\alpha}\omega_\varphi^{2r}(f,t)_p)^q\,\frac{1}{t}\,dt+\|f\|_p^q\right).
\end{aligned}$$

To prove the inverse part, we choose $2\leqslant\lambda\in N$, which will be determined later, and let $\{n_k\}_{k\in N}$ be a sequence of integers such that $\lambda^{k-1}\leqslant n_k^r<\lambda^k$ and

$$\|\varphi^{2r}P_{n_k,r}^{(2r)}(f,x)\|_p=\min_{\lambda^{k-1}\leqslant n^r<\lambda^k}\{\|\varphi^{2r}P_{n,r}^{(2r)}(f,x)\|_p\}. \tag{3.2}$$

We now recall the Peetre K-functional

$$K_{2r,\varphi}(f,t^{2r})_p=\inf_{g^{(2r-1)}\in A\cdot C_{loc}}(\|f-g\|_{L_p[0,\infty)}+t^{2r}\,\|\varphi^{2r}g^{(2r)}\|_{L_p[0,\infty)}),$$
$$1\leqslant p\leqslant\infty, \tag{3.3}$$

which was shown in [1, Chap. 2] to be equivalent to $\omega_\varphi^{2r}(f,t)_p$.

For $0<q<\infty$, we have

$$\begin{aligned}
\int_0^\infty (t^{-2\alpha}K_{2r,\varphi}(f,t^{2r})_p)^q\,\frac{dt}{t}&\leqslant\int_0^1 (t^{-2\alpha}K_{2r,\varphi}(f,t^{2r})_p)^q\,\frac{dt}{t}+M\,\|f\|_p^q\\
&\leqslant M\left(\sum_{k=0}^{\infty}(\lambda^{k\alpha/r}K_{2r,\varphi}(f,\lambda^{-k})_p)^q+\|f\|_p^q\right).
\end{aligned}$$

We fix $m\in N$ and let $U_k=\lambda^{k\alpha/r}K_{2r,\varphi}(P_{m,r}(f,x),\lambda^{-k})_p$, and obtain by using Theorem 9.3.2, Lemma 9.7.2 of [1], and Lemma 2.5,

$$\begin{aligned}
U_k&\leqslant\lambda^{k\alpha/r}\,\|P_{m,r}(f,x)-P_{n_{k+2},r}(P_{m,r}(f,\cdot),x)\|_p\\
&\quad+\lambda^{k(\alpha/r-1)}\,\|\varphi^{2r}P_{m,r}^{(2r)}(P_{n_{k+2},r}(f,\cdot),x)\|_p
\end{aligned}$$

$$\leqslant M\lambda^{k\alpha/r} K_{2r,\varphi}(P_{m,r}(f,x), n_{k+2}^{-r})_p + M\lambda^{k(\alpha/r-1)} \|f\|_p$$

$$+ M\lambda^{k(\alpha/r-1)} \|\varphi^{2r} P^{(2r)}_{n_{k+2},r}(f,x)\|_p$$

$$\leqslant M\lambda^{-\alpha/r} U_{k+1} + M\lambda^{k(\alpha/r-1)}(\|\varphi^{2r} P^{(2r)}_{n_{k+2},r}(f,x)\|_p + \|f\|_p)$$

$$\leqslant (M\lambda^{-\alpha/r})^j\, U_{k+j} + M\lambda^{k(\alpha/r-1)} \sum_{l=0}^{j-1} (M\lambda^{-1})^l$$

$$\times (\|\varphi^{2r} P^{(2r)}_{n_{k+2+l},r}(f,x)\|_p + \|f\|_p).$$

Since

$$U_{k+j} = \lambda^{(k+j)k\alpha/r} K_{2r,\varphi}(P_{m,r}(f,x), \lambda^{-k-j})_p$$
$$\leqslant \lambda^{(k+j)(\alpha/r-1)} \|\varphi^{2r} P^{(2r)}_{m,r}(f,x)\|_p.$$

Taking λ to be big enough, then we have

$$(M\lambda^{-\alpha/r})^l\, U_{k+j} \leqslant (M\lambda^{-1})^j\, \lambda^{k(\alpha/r-1)} \|\varphi^{2r} P^{(2r)}_{m,r}(f,x)\|_p \to 0, \qquad j \to \infty.$$

Hence

$$U_k \leqslant M\lambda^{k(\alpha/r-1)} \sum_{l=0}^{\infty} (M\lambda^{-1})^l (\|\varphi^{2r} P^{(2r)}_{n_{k+2+l},r}(f,x)\|_p + \|f\|_p).$$

Note that for $f \in L_p[0,\infty)$, $1 \leqslant p < \infty$, and $C[0,\infty)$ for $p = \infty$, we have

$$\|P_{m,r}(f,x) - f(x)\|_p \to 0, m \to \infty,$$

and therefore

$$\lambda^{k\alpha/r} K_{2r,\varphi}(f, \lambda^{-k})_p \leqslant M\lambda^{k(\alpha/r-1)} \sum_{l=0}^{\infty} (M\lambda^{-1})^l (\|\varphi^{2r} P^{(2r)}_{n_{k+2+l},r}(f,x)\|_p + \|f\|_p). \tag{3.4}$$

For $0 < q < \infty$, we choose $0 < \mu < \min\{1, q\}$, we have

$$\sum_{k=0}^{\infty} (\lambda^{k\alpha/r} K_{2r,\varphi}(f, \lambda^{-k})_p)^q$$

$$\leqslant M\|f\|_p^q + \sum_{k=0}^{\infty} (M\lambda^{k(\alpha/r-1)})^q \left\{ \sum_{l=0}^{\infty} [(M\lambda^{-1})^l \|\varphi^{2r} P^{(2r)}_{n_{k+2+l},r}(f,x)\|_p]^\mu \right\}^{q/\mu}.$$

Taking $\beta=\alpha/2$ and $\lambda>M^{2r/\alpha}$. Then the second term can be deduced by the Hölder inequality as

$$\left\{\sum_{l=0}^{\infty}\left[(M\lambda^{-1})^{l}\,\|\varphi^{2r}P^{(2r)}_{n_{k+2+l},\,r}(f,x)\|_p\right]^{\mu}\right\}^{q/\mu}$$

$$\leqslant\left\{\sum_{l=0}^{\infty}(n_{k+l+2}^{\beta-r}\,\|\varphi^{2r}P^{(2r)}_{n_{k+2+l},\,r}(f,x)\|_p)^q\right\}$$

$$\left\{\sum_{l=0}^{\infty}\left[(M\lambda^{-1})^{l}\,n_{k+l+2}^{r-\beta}\right]^{q\mu/(q-\mu)}\right\}^{q/\mu-1}$$

$$\leqslant C\sum_{l=k+2}^{\infty}(n_l^{\beta-r}\,\|\varphi^{2r}P^{(2r)}_{n_l,\,r}(f,x)\|_p)^q\lambda^{(k+2)(1-\beta/r)\,q}.$$

Then

$$\sum_{k=0}^{\infty}(\lambda^{k\alpha/r}K_{2r,\,\varphi}(f,\lambda^{-k})_p)^q$$

$$\leqslant C\sum_{k=0}^{\infty}\lambda^{k(\alpha/r-1)\,q+k(1-\beta/r)\,q}\sum_{l=k+2}^{\infty}(n_l^{\beta-r}\,\|\varphi^{2r}P^{(2r)}_{n_l,\,r}(f,x)\|_p)^q+M\,\|f\|_p^q$$

$$\leqslant C\sum_{l=2}^{\infty}(n_l^{\beta-r}\,\|\varphi^{2r}P^{(2r)}_{n_l,\,r}(f,x)\|_p)^q\sum_{k=0}^{l-2}\lambda^{qk(\alpha-\beta)/r}+M\,\|f\|_p^q$$

$$\leqslant C\sum_{l=2}^{\infty}(\lambda^{(l-1)(\beta/r-1)+(l-2)(\alpha-\beta)/r})^q\,\|\varphi^{2r}P^{(2r)}_{n_l,\,r}(f,x)\|_p^q+M\,\|f\|_p^q$$

$$\leqslant C\sum_{l=2}^{\infty}\sum_{\lambda^{l-1}\leqslant n^r\leqslant\lambda^l}(n^{\alpha-r}\,\|\varphi^{2r}P^{(2r)}_{n,\,r}(f,x)\|_p)^q\frac{1}{n}+M\,\|f\|_p^q.$$

Therefore, we have

$$\left\{\int_0^{\infty}(t^{-2\alpha}K_{2r,\,\varphi}(f,t^{2r})_p)^q\,\frac{dt}{t}\right\}^{1/q}$$

$$\leqslant C\left\{\left[\sum_{n=1}^{\infty}(n^{\alpha-r}\,\|\varphi^{2r}P^{(2r)}_{n,\,r}(f,x)\|_p)^q\,\frac{1}{n}\right]^{1/q}+\|f\|_p\right\}.$$

Then we complete the proof of Theorem 1.5.

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